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An Approximation Approach to Non-strictly Convex Quadratic Semi-infinite Programming

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Abstract. We present in this paper a numerical method for solving non-strictly-convex quadratic semi-infinite programming including linear semi-infinite programming. The proposed method transforms the problem into a series of strictly convex quadratic semi-infinite programming problems. Several convergence results and a numerical experiment are given.

Key words: Approximation, Convex quadratic semi-infinite programming, Duality, Linear semi-infinite programming

1. Introduction

This paper is concerned with a numerical method for solving convex quadratic semi-infinite programming problems with finitely many decision variables and infinitely many affine inequality constraints. For existing numerical methods of general semi-infinite programming, we refer the reader to a survey paper written by Reemtsen and Görner in Reemtsen and Rückmann (1998) and to relatively recent literatures such as Hettich and Kortanek (1993), Polak (1997), Shimizu et al. (1997), Goberna and López (1998), and Reemtsen and Rückmann (1998). Numerical methods of semi-infinite programming can be categorized into several groups: discretization methods, cutting plane methods, local reduction methods, nonsmooth optimization methods, exchange methods, interior point methods and others. Some of these are applicable to convex quadratic semi-infinite programming. An interior point approach was studied by Fang et al. (1994) for this class of problem.

On the other hand, the authors previously developed in Ito et al. (2000) a computational framework called dual parametrization for solving convex semi-infinite programming problems (see also Liu et al., 1999, 2001a). In this framework, a convex quadratic semi-infinite programming problem is reduced to the problem of finding a global solution (together with the corresponding multipliers) of some finite-dimensional nonlinear programming problem. A practical numerical method for finding such a global solution was developed in Liu et al. (2001b) for strictly convex quadratic semi-infinite programming. We present in this paper an approximation approach to the solution of non-strictly-convex quadratic semi-infinite programming including the linear case. The proposed method transforms the problem into a series of strictly convex quadratic semi-infinite programming problems, each of which is solved by a numerical technique develped in Liu et al. (2001b). Convergence results and a numerical experiment are given.

2. Convex Quadratic Semi-Infinite Programming and Duality

Consider the following convex quadratic semi-infinite programming problem:

$$\begin{aligned} \min_{x} \ & f(x) = \frac{1}{2}x^T Q x + b^T x \\ \text{subject to} \ & a(t)^T x \leqslant c(t) \quad \text{for } t \in T, \end{aligned} \tag{P}$$

where $x \in R^n$ is the decision variable; $0 \leqslant Q \in R^{n\times n}$, $b \in R^n$; $a : T \to R^n$ and $c : T \to R$ are continuously differentiable functions defined on a compact subset T of some Euclidean space. The order $Q \geqslant 0$ denotes that the matrix Q is positive semidefinite, while $Q > 0$ means that Q is a positive definite matrix. The index set T may be implicitly specified with a set of inequalities and/or equalities in some applications.

The Dorn-type dual form of problem (P) is

$$\begin{aligned} \max_{x,\Lambda} \ & -\frac{1}{2}x^T Q x - \int_T c(t)\, d\Lambda \\ \text{subject to} \ & Qx + b + \int_T a(t)\, d\Lambda = 0, \\ & \Lambda \geqslant 0, \end{aligned} \tag{D}$$

where the dual variable Λ is sought over the space of all finite signed regular Borel measures defined on the index set T, and the second constraint requires Λ to be a nonnegative (i.e., not signed) regular Borel measure. It should be noted that the variable x in problem (D) is only used to determine the feasible region and has nothing to do with the maximization itself. In fact, the dual objective has a constant value for each Λ within the feasible region.

We now assume that Slater's constraint qualification holds for the primal problem (P), i.e., there exists an $x \in R^n$ satisfying $a(t)^T x < c(t)$ for all $t \in T$. Then the strong duality holds, i.e., if there exists a solution to problem (P), then there also exists a solution pair to problem (D), and there is no duality gap between the primal and dual objectives at their solutions.

Due to the finite dimensionality of the primal variable x, we have the following important property, which is a consequence of Carathéodory's lemma.

PROPOSITION 1. *Suppose that problem* (*P*) *has a solution. Then the set of solution pairs to problem* (*D*) *necessarily contains a pair with a finite measure supported at no more than n points.*

Proof. See Theorem 12 of Ito et al. (2000). □

Our main concern is to find an optimal solution to the primal problem. When the dual problem is solved yielding a dual solution pair (x^*, Λ^*), this x^* does not necessarily solve the primal problem unless it is primal feasible and the pair (x^*, Λ^*) satisfies the so-called complementarity condition.

We now have the following properties on the converse duality.

PROPOSITION 2. *Let (x^*, Λ^*) be a solution pair of the dual problem (D).*

(i) If $Q > 0$, then x^* itself gives a solution to the primal problem (P).
(ii) Let ξ^* be a multiplier corresponding to the equality constraint of problem (D). Then ξ^* gives a solution to the primal problem (P).

Proof. (i) When $Q > 0$, the Lagrangian functional for problem (P) is strictly convex with respect to x. Then it can be shown that the unique x^* is primal feasible and (x^*, Λ^*) satisfies the complementarity condition. See Theorems 13 and 14 of Ito et al. (2000) for details. (ii) See Theorem 15 of Ito et al. (2000). □

3. Dual Parametrization

The first proposition in the last section suggests the search of a dual solution in a subset of the measure space, where each of its element is characterized by the location of a finite number of supporting points and the measures of these points. Let us fix the number of supporting points to k, where $k \leqslant n$ according to Proposition 1, and denote these points by t_i, $i = 1, 2, \ldots, k$, and their measures by λ_i, $i = 1, 2, \ldots, k$, respectively. Then the dual problem (D) can be equivalently written as:

$$\begin{aligned} &\max_{x, t_i, \lambda_i,\, i=1,2,\ldots,k} \quad -\frac{1}{2}x^T Q x - \sum_{i=1}^{k} \lambda_i\, c(t_i) \\ &\text{subject to} \quad Qx + b + \sum_{i=1}^{k} \lambda_i\, a(t_i) = 0, \\ &\qquad\qquad\quad t_i \in T, \quad \lambda_i \geqslant 0, \quad i = 1, 2, \ldots, k, \end{aligned} \tag{D$'$}$$

which is a finite-dimensional nonlinear programming problem. Unfortunately, it is not convex.

According to Proposition 1, the true number k^* of supporting points of the optimal discrete measure for a given problem lies between 0 and n, but it is not known a priori. We therefore need to make a guess of k when solving the parametrized dual problem (D$'$). The most natural choice is $k = n$. However, a smaller k is better from a computational perspective as far as $k \geqslant k^*$ because a larger k increases the number of decision variables. When $n > 1$, the minimal number k^* is usually less (often much less) than n. Hence it is, especially for large-scale problems, advantageous to start with a small k and to increase it gradually while solving problem (D′) for each k until some convergence criterion is satisfied.

It should be noted that we need to find a global solution of the parametrized dual problem (D′), which can be highly nonlinear and multi-modal due to the nonlinearity and nonconvexity of the functions c and a. A practical numerical procedure for finding a global solution of problem (D′) was developed in Liu et al. (2001b) for strictly convex quadratic semi-infinite programming. Based on the fact that problem (D′) is nonlinear only with respect to t_i's, it proceeds as follows: (1) first choose an integer k, fix k points $(t_1, t_2, \ldots, t_k)$ and solve the quadratic programming problem

$$\begin{aligned} \max_{x,\lambda_i,\, i=1,2,\ldots,k} \quad & -\frac{1}{2}x^T Qx - \sum_{i=1}^{k} \lambda_i\, c(t_i) \\ \text{subject to} \quad & Qx + b + \sum_{i=1}^{k} \lambda_i\, a(t_i) = 0, \\ & \lambda_i \geqslant 0, \quad i = 1, 2, \ldots, k \end{aligned}$$

and/or the corresponding primal form

$$\begin{aligned} & \min_x \ \frac{1}{2}x^T Qx + b^T x \\ & \text{subject to} \quad a(t_i)^T x \leqslant c(t_i), \quad i = 1, 2, \ldots, k \end{aligned}$$

for the fixed k and t_i's; (2) then increase k and update t_i's, and repeat the process for finding an approximate solution of problem (D′) until some stop criterion is satisfied; and (3) finally move on to the nonlinear search for an accurate global solution starting from the approximate solution.

This procedure is only applicable to the strictly convex quadratic case since it is based on property (i) of Proposition 2. A similar procedure based on property (ii) can be constructed for the general convex quadratic case. However, we need to solve the parametrized dual (D′) instead of (D), and it is not a numerically stable task to find an optimal multiplier for the equality constraint. We therefore consider in the next section another approach for the numerical solution of non-strictly convex quadratic semi-infinite programming including the linear case.

4. ε-Approximation for the Non-Strictly Convex Case

For a given $\varepsilon > 0$, let us approximate problem (P) by

$$\begin{aligned} & \min_x \ f_\varepsilon(x) = \frac{1}{2}x^T Q_\varepsilon x + b^T x \\ & \text{subject to} \quad a(t)^T x \leqslant c(t) \quad \text{for } t \in T, \end{aligned} \tag{P_ε}$$

where $Q_\varepsilon = Q + \varepsilon I$ (I is the identity matrix).

PROPOSITION 3. *Suppose that problem* (P) *has a solution. Then, for any* $\varepsilon > 0$, *problem* (P_ε) *has a unique solution* x^*_ε *such that*

$$x^*_\varepsilon \to x^* \quad \textit{as} \quad \varepsilon \to 0,$$

where x^* *is a unique minimum-norm solution of problem* (P), *i.e.*,

$$\|x^*\| = \min \{ \|x\| \mid x \textit{ solves problem } (P) \}.$$

Proof. Since problem (P) has a solution, the feasible region of problem (P) is a nonempty closed convex set. It is then clear that problem (P_ε) has a unique solution, denoted above by x^*_ε, since $Q_\varepsilon > 0$. Let $S(\mathrm{P})$ be the solution set of problem (P). Since $S(\mathrm{P})$ is a nonempty closed convex set in R^n, there exists a unique point $x^* \in S(\mathrm{P})$ such that

$$\|x^*\| = \min_{x \in S(\mathrm{P})} \|x\|.$$

We now prove that x^*_ε converges to x^* as $\varepsilon \to 0$. Note that, for any $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{2}\varepsilon\|x^*_\varepsilon\|^2 &= f_\varepsilon(x^*_\varepsilon) - f(x^*_\varepsilon) \\ &\leqslant f_\varepsilon(x^*) - f(x^*) \\ &= \frac{1}{2}\varepsilon\|x^*\|^2, \end{aligned}$$

i.e.,

$$\|x^*_\varepsilon\| \leqslant \|x^*\|. \tag{1}$$

Hence, $\{ x^*_\varepsilon \mid \varepsilon > 0 \}$ is bounded by $\|x^*\|$. Let $\{\varepsilon_k\}$ be any sequence such that $\varepsilon_k > 0$, $k = 1, 2, \ldots$, and

$$\varepsilon_k \to 0 \quad (k \to \infty). \tag{2}$$

Let $\{x^*_{\varepsilon_{k_i}}\}$ be any convergent subsequence of $\{x^*_{\varepsilon_k}\}$ such that

$$x^*_{\varepsilon_{k_i}} \to \bar{x} \quad (i \to \infty)$$

for some $\bar{x} \in R^n$. It is clear that $\bar{x}$ is feasible and

$$\begin{aligned} f(x^*) &\leqslant f(\bar{x}) \\ &= \lim_{i\to\infty} f(x^*_{\varepsilon_{k_i}}) \\ &= \lim_{i\to\infty}\Big(f(x^*_{\varepsilon_{k_i}}) + \frac{\varepsilon_{k_i}}{2}\|x^*_{\varepsilon_{k_i}}\|^2\Big) \\ &= \lim_{i\to\infty} f_{\varepsilon_{k_i}}(x^*_{\varepsilon_{k_i}}) \\ &\leqslant \lim_{i\to\infty} f_{\varepsilon_{k_i}}(x^*) \\ &= \lim_{i\to\infty}\Big(f(x^*) + \frac{\varepsilon_{k_i}}{2}\|x^*\|^2\Big) \\ &= f(x^*). \end{aligned}$$

Thus, $\bar{x}$ is a solution of problem (P). From the definition of x^*, we have

$$\|x^*\| \leqslant \|\bar{x}\|,$$

and inequality (1) means

$$\|\bar{x}\| = \lim_{i\to\infty} \|x^*_{\varepsilon_{k_i}}\| \leqslant \|x^*\|.$$

Thus $\|\bar{x}\| = \|x^*\|$, and hence $\bar{x} = x^*$ by the uniqueness of the minimum-norm point in $S(\mathrm{P})$. Therefore, we have

$$\lim_{i\to\infty} x^*_{\varepsilon_{k_i}} = x^*.$$

Since any convergent subsequence of the bounded sequence $\{x^*_{\varepsilon_k}\}$ converges to x^*, we see that

$$\lim_{k\to\infty} x^*_{\varepsilon_k} = x^*,$$

which further implies

$$x^*_\varepsilon \to x^* \quad (\varepsilon \to 0)$$

since $\{\varepsilon_k\}$ is an arbitrary sequence satisfying (2). □

This proposition can be slightly extended. Let x_0 be any given point in R^n. Consider the quadratic function

$$f_{\varepsilon,x_0}(x) = \frac{1}{2}x^T Q_\varepsilon x + b^T_{\varepsilon,x_0} x,$$

where $b_{\varepsilon,x_0} = b - \varepsilon x_0$. Note that f_{ε,x_0} is obtained by taking away the constant $(1/2)\varepsilon\|x_0\|^2$ from the quadratic function

$$\frac{1}{2}x^T Q x + b^T x + \frac{1}{2}\varepsilon\|x - x_0\|^2.$$

Defining the strictly convex problem

$$\begin{aligned} &\min_x \ f_{\varepsilon,x_0}(x) \\ &\text{subject to} \quad a(t)^T x \leqslant c(t) \quad \text{for } t \in T, \end{aligned} \tag{P_{ε,x_0}}$$

we have the following proposition.

PROPOSITION 4. *Suppose that problem* (P) *has a solution. Then, for any* $\varepsilon > 0$, *problem* (P_{ε,x_0}) *has a unique solution* x^*_{ε,x_0} *such that*

$$x^*_{\varepsilon,x_0} \to x^*_{x_0} \quad as \quad \varepsilon \to 0,$$

where $x^*_{x_0}$ *is a solution of problem* (P) *that is closest to* x_0, *i.e.*,

$$\|x^*_{x_0} - x_0\| = \min\{\, \|x - x_0\| \mid x \text{ solves problem } (P)\,\}.$$

Proof. The proof is similar to that of Proposition 3 and is omitted. □

5. Algorithm and the Solution Sequence

Our algorithm for solving problem (P) is then described as follows.

Algorithm:

(1°) Start with some positive ε.

(2°) Find a unique minimizer x^*_ε of problem (P_ε) by solving its parametrized dual:

$$\begin{aligned} &\max_{x,t_i,\lambda_i,\, i=1,2,\dots,k} \ -\frac{1}{2}x^T Q_\varepsilon x - \sum_{i=1}^{k} \lambda_i\, c(t_i) \\ &\text{subject to} \quad Q_\varepsilon x + b + \sum_{i=1}^{k} \lambda_i\, a(t_i) = 0, \\ &\qquad t_i \in T, \quad \lambda_i \geqslant 0, \quad i = 1, 2, \dots, k \end{aligned} \tag{D'_ε}$$

as was explained in Section 3.

(3°) Decrease ε and continue until some convergence is observed.

Let x^* be as in Proposition 3. We have the following properties for the solution sequence $\{x^*_\varepsilon\}$ of problem (P_ε).

PROPOSITION 5. $\|x^*_\varepsilon\| \leqslant \|x^*\|$ *for any* $\varepsilon > 0$, *and if* $\|x^*_\varepsilon\| = \|x^*\|$ *for some* ε, *then* x^*_ε *gives a solution to problem* (P).

Proof. We have

$$f(x^*) \leqslant f(x^*_\varepsilon)$$

and

$$f(x_\varepsilon^*) + \frac{1}{2}\varepsilon\|x_\varepsilon^*\|^2 \leqslant f(x^*) + \frac{1}{2}\varepsilon\|x^*\|^2.$$

The latter follows from $f_\varepsilon(x_\varepsilon^*) \leqslant f_\varepsilon(x^*)$. Adding these inequalities yields $\|x_\varepsilon^*\| \leqslant \|x^*\|$. If $\|x_\varepsilon^*\| = \|x^*\|$ for some ε, the second inequality gives $f(x_\varepsilon^*) \leqslant f(x^*)$. Together with the first inequality, we obtain $f(x_\varepsilon^*) = f(x^*)$. □

PROPOSITION 6. *$\|x_\varepsilon^*\|$ is nondecreasing and $f(x_\varepsilon^*)$ is nonincreasing as ε tends to zero.*

Proof. Let $\varepsilon_1 > \varepsilon_2 > 0$. It follows from $f_{\varepsilon_1}(x_{\varepsilon_1}^*) \leqslant f_{\varepsilon_1}(x_{\varepsilon_2}^*)$ and $f_{\varepsilon_2}(x_{\varepsilon_2}^*) \leqslant f_{\varepsilon_2}(x_{\varepsilon_1}^*)$ that

$$f(x_{\varepsilon_1}^*) + \frac{1}{2}\varepsilon_1\|x_{\varepsilon_1}^*\|^2 \leqslant f(x_{\varepsilon_2}^*) + \frac{1}{2}\varepsilon_1\|x_{\varepsilon_2}^*\|^2 \tag{3}$$

and

$$f(x_{\varepsilon_2}^*) + \frac{1}{2}\varepsilon_2\|x_{\varepsilon_2}^*\|^2 \leqslant f(x_{\varepsilon_1}^*) + \frac{1}{2}\varepsilon_2\|x_{\varepsilon_1}^*\|^2. \tag{4}$$

Adding these inequalities leads to $\|x_{\varepsilon_1}^*\| \leqslant \|x_{\varepsilon_2}^*\|$. Similarly, by adding the inequalities multiplied, respectively, by ε_2 and ε_1, we obtain $f(x_{\varepsilon_2}^*) \leqslant f(x_{\varepsilon_1}^*)$. □

PROPOSITION 7. *Let $\varepsilon_1 > \varepsilon_2 > 0$. If one of the following three conditions is satisfied, then the other two always hold*:

$$f(x_{\varepsilon_1}^*) = f(x_{\varepsilon_2}^*), \quad \|x_{\varepsilon_1}^*\| = \|x_{\varepsilon_2}^*\|, \quad x_{\varepsilon_1}^* = x_{\varepsilon_2}^*. \tag{5}$$

Proof. When either one of the first two conditions holds, the other follows from inequalities (3) and (4). Then we obtain $f_{\varepsilon_1}(x_{\varepsilon_1}^*) = f_{\varepsilon_1}(x_{\varepsilon_2}^*)$ (respectively, $f_{\varepsilon_2}(x_{\varepsilon_1}^*) = f_{\varepsilon_2}(x_{\varepsilon_2}^*)$), which leads to $x_{\varepsilon_1}^* = x_{\varepsilon_2}^*$ since f_{ε_1} (respectively, f_{ε_2}) is strictly convex. □

A point $x_\infty^* = \lim_{\varepsilon\to\infty} x_\varepsilon^*$ always exists as a minimum-norm feasible solution. We have

$$\begin{aligned} f_\varepsilon(x) &= \frac{1}{2}(x + Q_\varepsilon^{-1}b)^T Q_\varepsilon (x + Q_\varepsilon^{-1}b) - \frac{1}{2}b^T Q_\varepsilon^{-1} b \\ &= \frac{1}{2}\|x + Q_\varepsilon^{-1}b\|_{Q_\varepsilon}^2 - \frac{1}{2}b^T Q_\varepsilon^{-1} b, \end{aligned}$$

where $\|\cdot\|_{Q_\varepsilon}$ denotes a norm induced by the positive definite matrix Q_ε, i.e., $\|x\|_{Q_\varepsilon} := \sqrt{x^T Q_\varepsilon x}$. Hence, for each $\varepsilon > 0$, x_ε^* is the unique feasible solution that is closest to $-Q_\varepsilon^{-1}b$ in this variable metric. When ε varies from infinity to zero, the reference point $-Q_\varepsilon^{-1}b$ forms a continuous trajectory starting at the origin. The reference path converges to a minimum-norm (in the ordinary sense of Euclidean norm) solution of $Qx = b$, if it has a solution, or diverges otherwise. Especially, when $Q = 0$, the divergent path is a half straight line passing through $-b$.

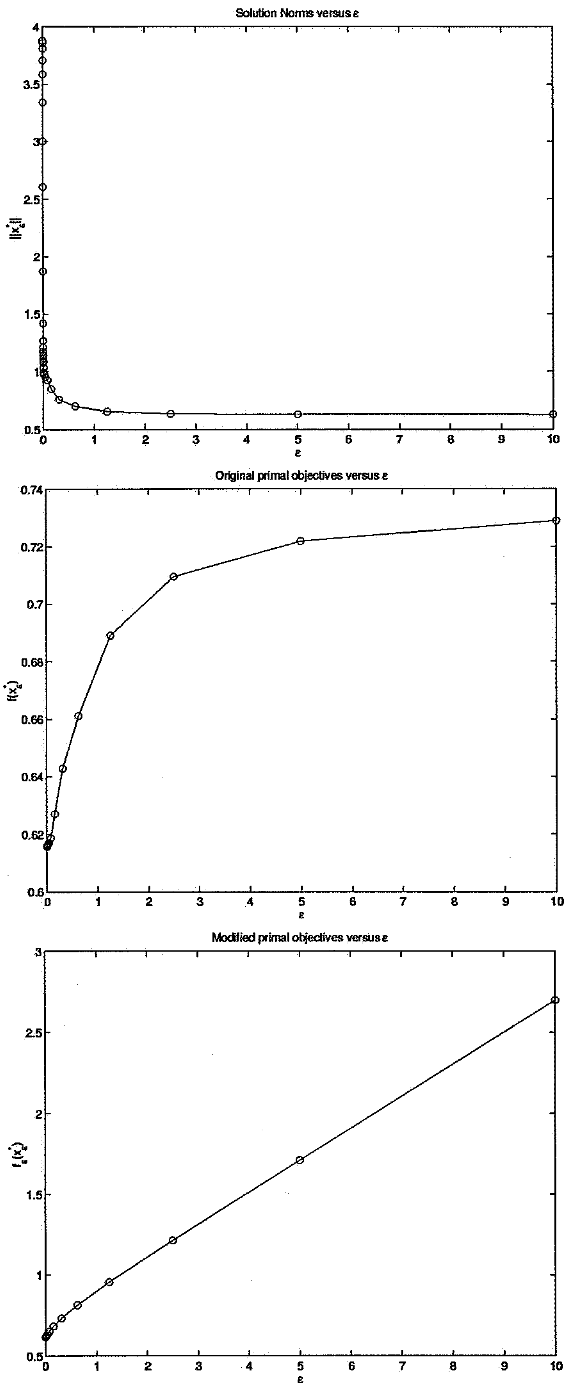

Figure 1. Numerical results.

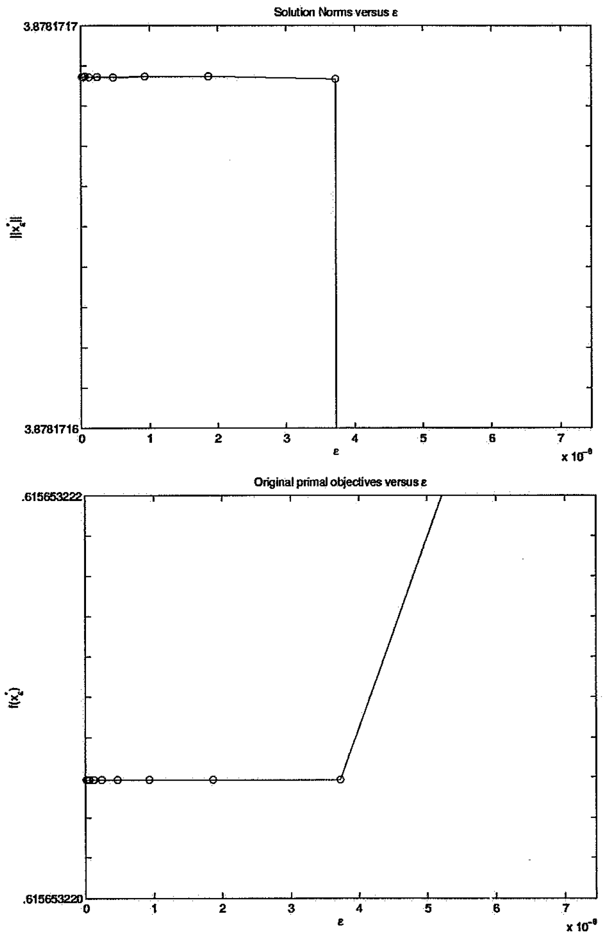

Figure 2. Enlarged graphs of $\|x_\varepsilon^*\|$ and $f(x_\varepsilon^*)$.

6. Numerical Experiment

As an example of linear semi-infinite programming, let us consider the following one-sided L_1 approximation problem:

$$\min_x \; f(x) = \sum_{i=1}^{n} \frac{x_i}{i}$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_i t^{i-1} \geqslant \tan t \quad \text{for } t \in [0, 1],$$

where the tangent curve is approximated by a polynomial of degree $n - 1$ over the interval [0, 1]. This problem can be written in the form of problem (P), where the

Table 1. Numerical results

ε	$\|x_\varepsilon^*\|$	$f(x_\varepsilon^*)$	$f_\varepsilon(x_\varepsilon^*)$
10.000 000 000 000	0.627 498 223	0.728 939 324	2.697 719 656
5.000 000 000 000	0.629 180 895	0.721 860 265	1.711 527 001
2.500 000 000 000	0.634 964 254	0.709 557 769	1.213 533 122
1.250 000 000 000	0.653 799 651	0.689 110 836	0.956 269 651
0.625 000 000 000	0.700 552 542	0.661 147 117	0.814 524 242
0.312 500 000 000	0.757 616 679	0.642 883 258	0.732 568 284
0.156 250 000 000	0.848 911 930	0.627 022 976	0.683 323 878
0.078 125 000 000	0.924 611 110	0.618 603 162	0.651 997 859
0.039 062 500 000	0.953 270 521	0.617 068 387	0.634 816 886
0.019 531 250 000	0.974 319 074	0.616 487 221	0.625 757 715
0.009 765 625 000	0.993 851 686	0.616 220 132	0.621 043 089
0.004 882 812 500	1.034 042 613	0.615 950 261	0.618 560 721
0.002 441 406 250	1.079 203 447	0.615 759 463	0.617 181 191
0.001 220 703 125	1.092 331 543	0.615 734 964	0.616 463 228
0.000 610 351 563	1.111 971 134	0.615 716 404	0.616 093 748
0.000 305 175 781	1.139 406 093	0.615 703 115	0.615 901 212
0.000 152 587 891	1.172 807 405	0.615 694 838	0.615 799 779
0.000 076 293 945	1.209 500 763	0.615 690 158	0.615 745 963
0.000 038 146 973	1.267 613 667	0.615 686 359	0.615 717 007
0.000 019 073 486	1.419 053 066	0.615 681 139	0.615 700 343
0.000 009 536 743	1.873 408 608	0.615 671 616	0.615 688 351
0.000 004 768 372	2.606 464 089	0.615 659 996	0.615 676 194
0.000 002 384 186	3.003 717 616	0.615 656 153	0.615 666 908
0.000 001 192 093	3.341 549 958	0.615 654 267	0.615 660 922
0.000 000 596 046	3.585 659 260	0.615 653 527	0.615 657 358
0.000 000 298 023	3.706 558 410	0.615 653 330	0.615 655 377
0.000 000 149 012	3.811 485 453	0.615 653 240	0.615 654 323
0.000 000 074 506	3.858 058 741	0.615 653 224	0.615 653 779
0.000 000 037 253	3.878 171 687	0.615 653 221	0.615 653 501
0.000 000 018 626	3.878 171 687	0.615 653 221	0.615 653 361
0.000 000 009 313	3.878 171 687	0.615 653 221	0.615 653 291
0.000 000 004 657	3.878 171 687	0.615 653 221	0.615 653 256
0.000 000 002 328	3.878 171 687	0.615 653 221	0.615 653 238
0.000 000 001 164	3.878 171 687	0.615 653 221	0.615 653 229
0.000 000 000 582	3.878 171 687	0.615 653 221	0.615 653 225
0.000 000 000 291	3.878 171 687	0.615 653 221	0.615 653 223
0.000 000 000 146	3.878 171 687	0.615 653 221	0.615 653 222

coefficients are given by

$$Q = 0, \quad b = \begin{pmatrix} 1 \\ 1/2 \\ \vdots \\ 1/n \end{pmatrix}, \quad a(t) = -\begin{pmatrix} 1 \\ t \\ \vdots \\ t^{n-1} \end{pmatrix}, \quad c(t) = -\tan t.$$

We applied our algorithm described in the previous section to this example for $n = 8$. The experiment was done in the Matlab environment with the Optimization Toolbox. We started the iteration with $\varepsilon = 10$ and continued until it reaches to the order of 10^{-10}. For each ε, the norm, $\|x_\varepsilon^*\|$, the objective function values, $f(x_\varepsilon^*)$ and $f_\varepsilon(x_\varepsilon^*)$, are shown in Figure 1 and in Table 1. Enlarged graphs of $\|x_\varepsilon^*\|$ and $f(x_\varepsilon^*)$ are shown in Figure 2 for small ε's. As seen in Table 1 and in Figure 2, the finite termination is reached at some ε of the order 10^{-8}. For smaller ε's, the values $\|x_\varepsilon^*\|$ and $f(x_\varepsilon^*)$ remain constant while $f_\varepsilon(x_\varepsilon^*)$ continues decreasing.

7. Concluding Remarks

We have proposed in this paper an ε-approximation approach to the numerical solution of non-strictly convex quadratic semi-infinite programming including the linear case. This approach converts the problem into a sequence of strictly convex quadratic semi-infinite programming problems, each of which is solved by the dual parametrization technique developed in Liu et al. (2001b). As was observed in the numerical experiment, we can expect to obtain an exact solution to the original problem at a finite ε in some cases.

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